De Sitter Momentum Space

Based on arXiv:2601.15228
& long paper in preparation

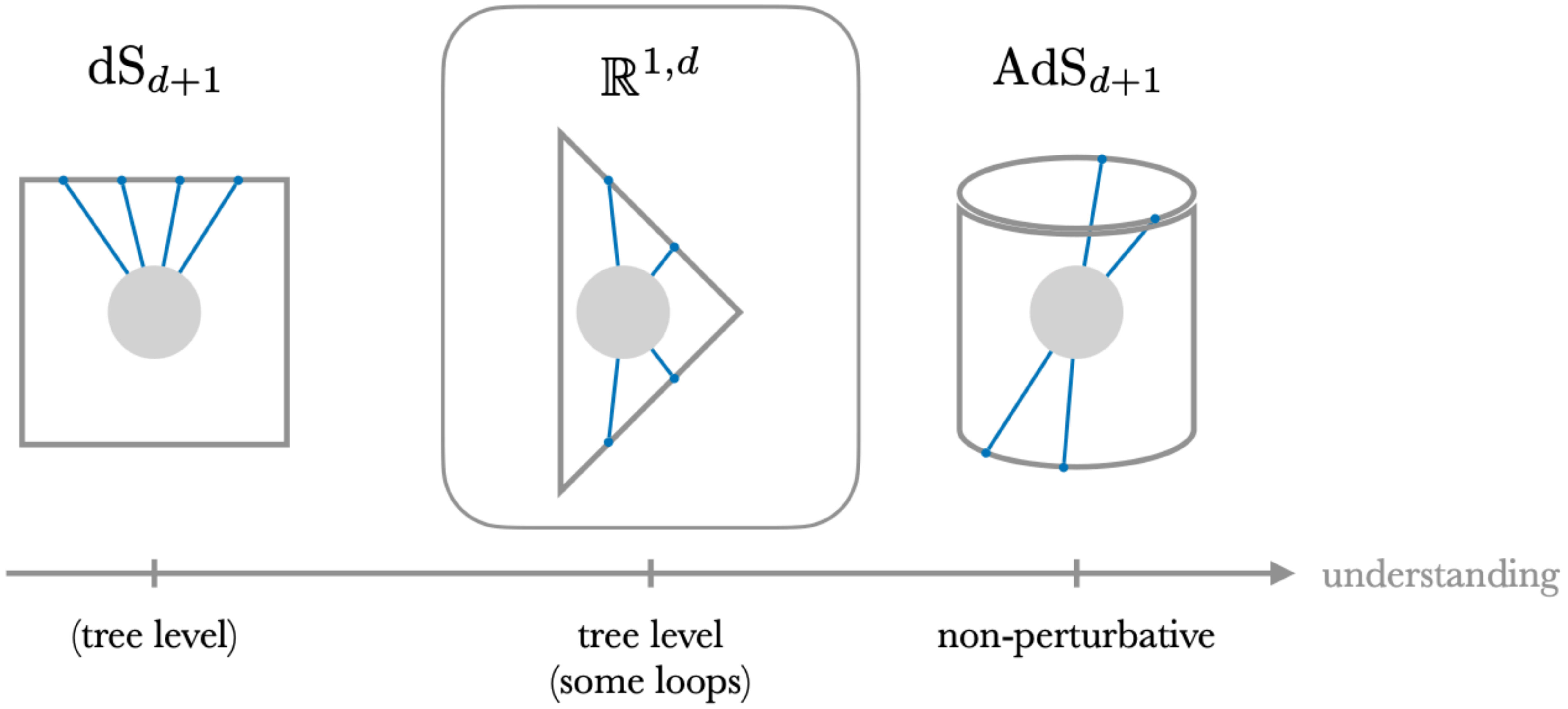
Arthur Poisson
work with N. Belrhali, S. Renaux-Petel & D. Werth

IAP, 23/02/2026

Motivation

- de Sitter spacetime is the simplest model of an accelerating universe.
- Spontaneous particle production in inflation allowing to probe very high energy processes.
- The least understood of three maximally symmetric spacetimes:

- No energy conservation: nested time integral even at tree level.
- Mode functions = special functions
- No dS/CFT correspondence



Momentum Space in QFT

- Time integrals reflects the lack of proper momentum space.
- Observable in QFT = vacuum correlation functions defined through path integral

Spacetime $\mathcal{M}$

$$\langle\Omega|\mathcal{O}(\boxed{X_1})\dots\mathcal{O}(\boxed{X_n})|\Omega\rangle = \int \mathcal{D}\varphi\mathcal{O}(X_1)\dots\mathcal{O}(X_n)e^{iS[\varphi]}$$

Wick rotation
$\mathcal{M} \to \mathcal{M}_E$

Definition of the vacuum
= time axis tilted in the
complex plane

The functional $S[\varphi]$ converges
= φ is square integrable

- If $\mathcal{M}$ has some symmetries, momentum space = basis of L^2 found by representing the symmetry group.

Outline

- Warm up: Minkowski Momentum Space
  - Review of the construction in Minkowski space in a dS-like way
- de Sitter Momentum Space
  - Construction of the Kontorovich-Lebedev-Fourier space (KLF)
  - Example of the conformal two point function
- QFT in momentum space: Perturbation Theory
  - Path integral in KLF space
  - Free theory and KLF propagators
  - Interaction vertices and examples: single exchange & loop integral

Outline

- Warm up: Minkowski Momentum Space
  - Review of the construction in Minkowski space is a dS-like way
- de Sitter Momentum Space
  - Construction of the Kontorovich-Lebedev-Fourier space (KLF)
  - Example of the conformal two point function
- QFT in momentum space: Perturbation Theory
  - Path integral in KLF space
  - Free theory and KLF propagators
  - Interaction vertices and examples: single exchange & loop integral

Warm-up: Minkowski Momentum Space

Goal: find a basis of $\mathrm{L}^2\left[\mathbb{M}^{d+1}\right]$

Idea: Representations of the symmetry group on this space

- Symmetry group = Poincaré: $\mathbb{R}^{d+1} \rtimes \mathrm{SO}(1,d)$

$P_\mu = i\partial_\mu$

Space & Time translations

Abelian subgroup

$[P_\mu, P_\nu] = 0$

$J_{\mu\nu} = i x_\mu \partial_\mu - i x_\nu \partial_\mu$

Rotation & Boosts

Lorentz Lie algebra

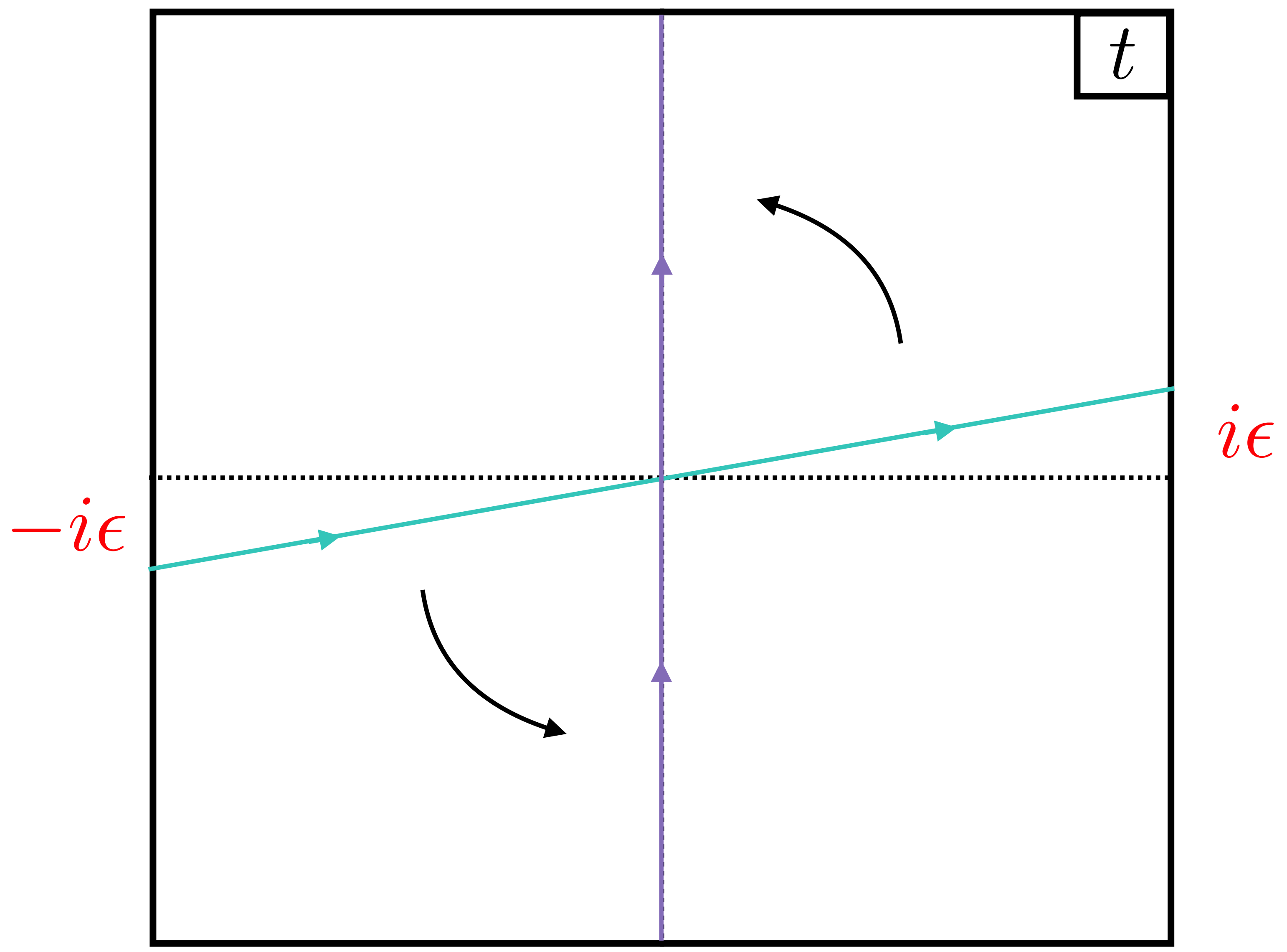
$[J_{\mu\nu}, J_{\rho\sigma}] = \#$

Lorentz vector

$[J_{\rho\sigma}, P_\mu] = 2i\eta_{\mu[\rho}P_{\sigma]}$

- Casimir operator: $\mathcal{C} = -P^\mu P_\mu \sim \Box_{\mathbb{M}}$ $\implies$ Commutes with all the algebra
- Schur Lemma: the eigenvalues of $\mathcal{C}$ label the Poincaré irreducible representations.
- Basis of the representation space = common eigenbasis to $\mathcal{C}$ and P_i

Warm-up: Minkowski Momentum Space



$\mathcal{C}$ is only self-adjoint on the Wick rotated axis $z = it$

$$\mathbb{M}^{d+1} \to \mathbb{R}^{d+1} \quad \text{and} \quad \Box_{\mathbb{M}} \to \Box^E_{\mathbb{M}} = \partial_z^2 + \partial_i^2$$

We define the Minkowski harmonic function as:

$$\begin{cases} \Box^E_{\mathbb{M}} \Phi^M_{\boldsymbol{p}}(z, \boldsymbol{x}) & = & M^2 \Phi^M_{\boldsymbol{p}}(z, \boldsymbol{x}) \\ \partial_i \Phi^M_{\boldsymbol{p}}(z, \boldsymbol{x}) & = & -ip_i \Phi^M_{\boldsymbol{p}}(z, \boldsymbol{x}) \end{cases}$$

Consistent solution with in-out vacuum boundary conditions:

$$\Phi^M_{\boldsymbol{p}}(z, \boldsymbol{x}) = e^{-i\boldsymbol{x}\cdot\boldsymbol{p}} \frac{e^{-\sqrt{\boldsymbol{p}^2 + M^2}|z|}}{2\sqrt{\boldsymbol{p}^2 + M^2}}$$

Warm-up: Minkowski Momentum Space

We found the Minkowski harmonic function $\Phi^M_{\boldsymbol{p}}(z,\boldsymbol{x}) = e^{-i\boldsymbol{x}\cdot\boldsymbol{p}}\dfrac{e^{-\sqrt{\boldsymbol{p}^2+M^2}|z|}}{2\sqrt{\boldsymbol{p}^2+M^2}}$

Equivalent to d+1 dimensional Fourier:

$$\boxed{f(z,\mathbf{x})} = \int_0^\infty \mathrm{d}M^2 \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d}\Phi^M_{\boldsymbol{p}}(z,\boldsymbol{x}) f^{(M)}_{\mathbf{p}} = \int \frac{\mathrm{d}^{d+1}\boldsymbol{p}\mathrm{d}\omega}{(2\pi)^d} e^{-i\mathbf{x}\cdot\mathbf{p}-iz\omega}\boxed{\hat{f}(\omega,\mathbf{p})}$$

L^2 function

"On-shell" expansion

Obtained by explicitly diagonalising $\mathcal{C}$ and P_i

"Off-shell" expansion

Obtained by explicitly diagonalising P_0 and P_i

$$\int_0^\infty \mathrm{d}M^2 \frac{f^{(M)}_{\boldsymbol{p}}}{M^2+\omega^2+\boldsymbol{p}^2}$$

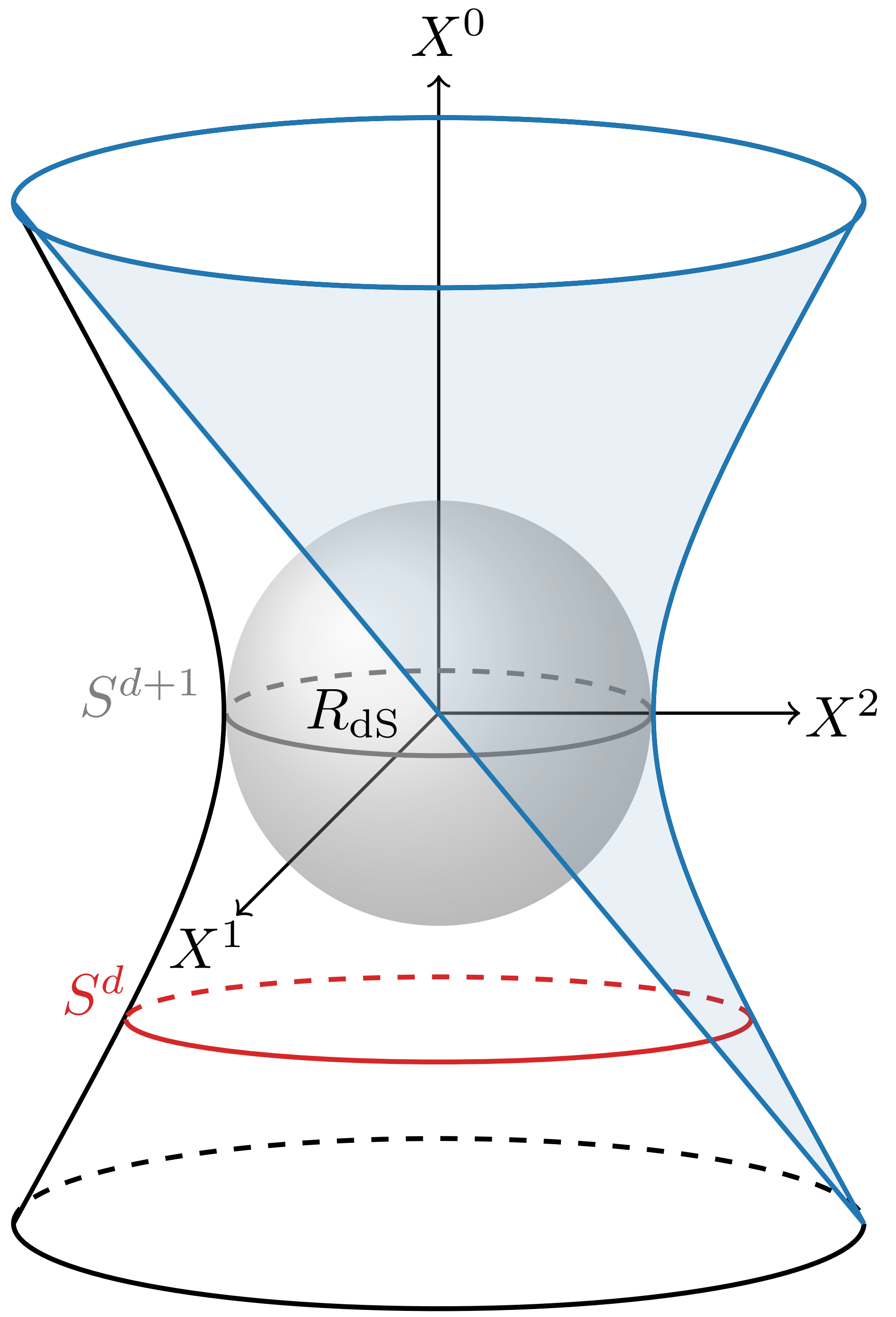
Related by a dispersion relation: $\omega_p^2 = M^2 + \boldsymbol{p}^2$

This is possible because $\mathcal{C}$ is part of an abelian sub-algebra.

Outline

- Warm up: Minkowski Momentum Space
  - Review of the construction in Minkowski space is a dS-like way
- de Sitter Momentum Space
  - Construction of the Kontorovich-Lebedev-Fourier space (KLF)
  - Example of the conformal two point function
- QFT in momentum space: Perturbation Theory
  - Path integral in KLF space
  - Free theory and KLF propagators
  - Interaction vertices and examples: single exchange & loop integral

De Sitter Space-Time: Basics



$\mathrm{dS}_2 \subset \mathbb{M}^3$, illustration

d+1 dimensional de Sitter is defined as an hyperboloid embedded in $\mathbb{M}^{d+2}$:

$$\eta_{AB} X^A X^B = -\frac{1}{H^2} \qquad (A,\ B = 0, 1, ..., d+1)$$

FLRW slicing = Poincaré coordinates:

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\tau^2 + \mathrm{d}\boldsymbol{x}^2}{(H\tau)^2} \quad \text{with} \quad a(\tau) = -\frac{1}{H\tau}$$

$\tau \in (-\infty, 0)$, conformal time

Constant τ hyper-surfaces = Euclidean planes $\mathbb{R}^d$

de Sitter Representation Theory (1/3)

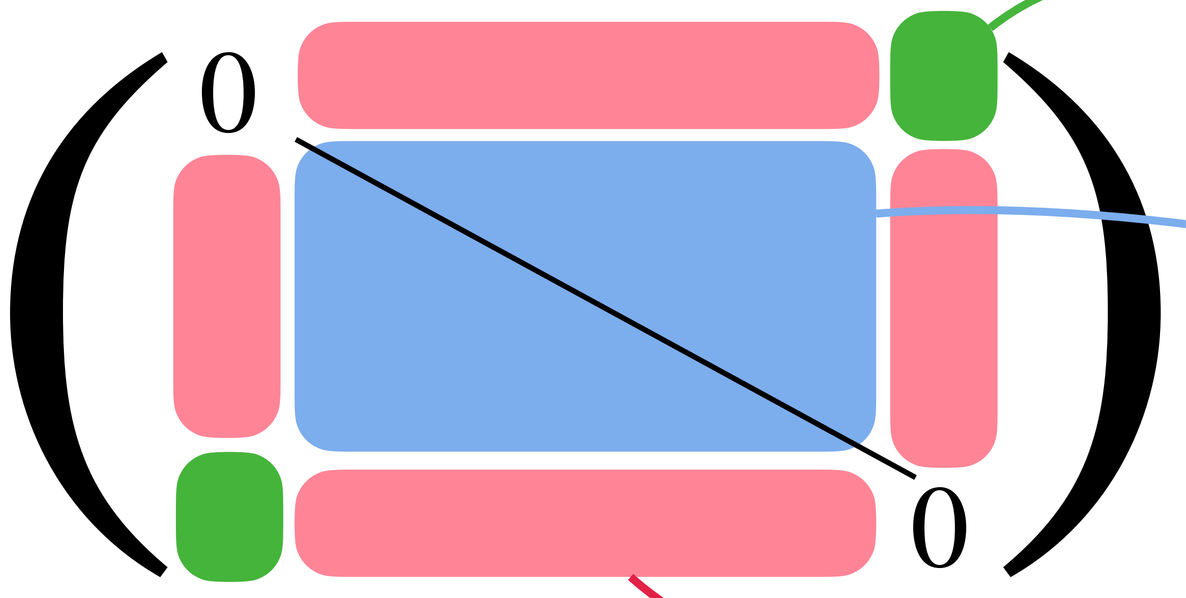
Goal: find a basis of $\mathrm{L}^2\left[\mathrm{dS}_{d+1}\right]$

Idea: Representations of the symmetry group on this space

- Symmetry group = d+2 dimensional Lorentz:

  $$\mathrm{SO}(1, d+1)$$

$$J_{AB} =$$



- Lorentz algebra: $[J_{AB}, J_{CD}] = \#$
- Casimir operator: $\mathcal{C} = -\frac{1}{2} J^{AB} J_{AB} \sim \Box_{\mathrm{dS}}$ $\Longrightarrow$ Commutes with all the algebra

$$= D(D - id) - P_i K^i - \frac{1}{2} M_{ij}^2$$

$\Longrightarrow$ Not part of an abelian sub-algebra

de Sitter Representation Theory (2/3)

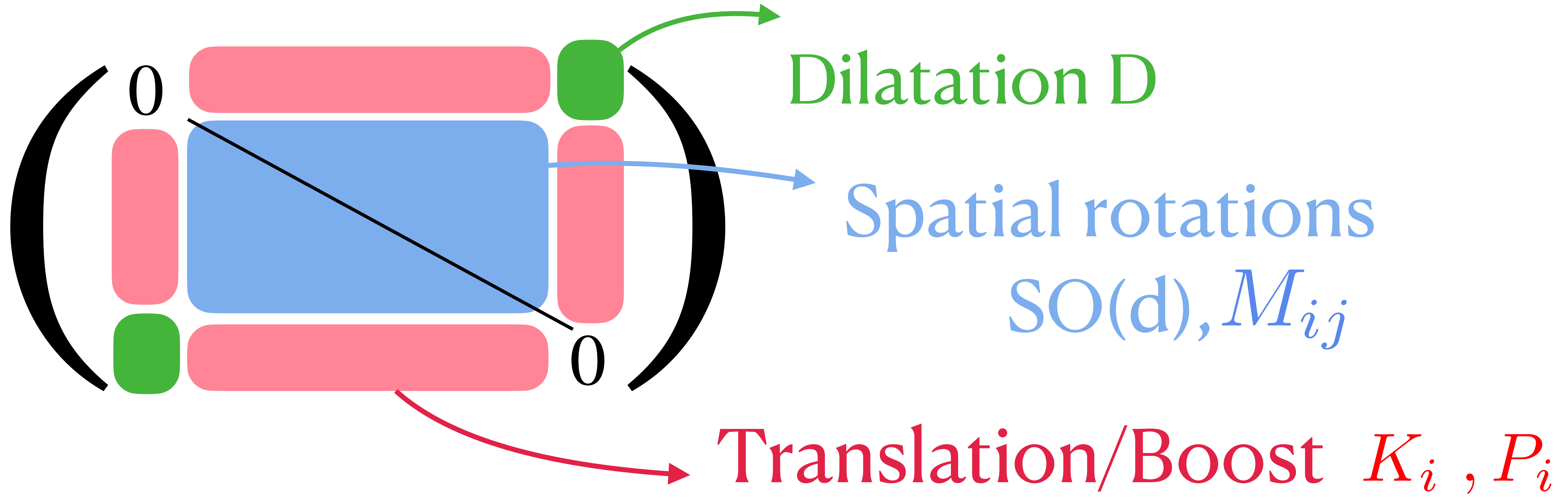
Goal: find a basis of $\mathrm{L}^2\left[\mathrm{dS}_{d+1}\right]$

Idea: Representations of the symmetry group on this space

- Symmetry group = d+2 dimensional Lorentz:

  $$\mathrm{SO}(1, d+1)$$

$J_{AB} =$

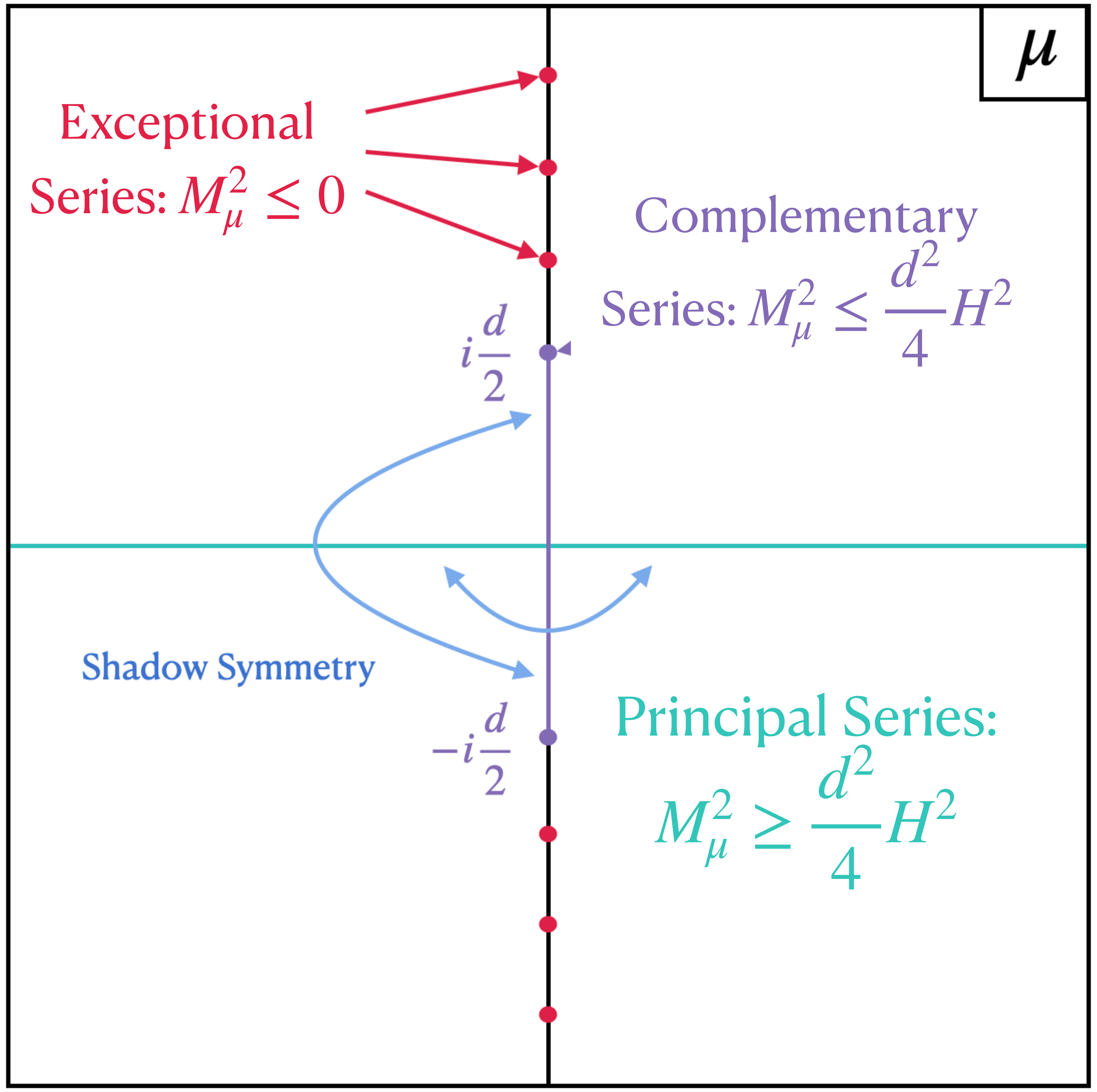


- Maximal set of commuting generators:

$$\mathcal{C} = -\frac{1}{2} J^{AB} J_{AB} \sim \Box_{\mathrm{dS}} \quad \text{and} \quad P_i \sim i\partial_i$$

de Sitter Representation Theory (3/3)

- Diagonalising the Casimir: Classifying the possible particle masses.



$$C\,|\mu\rangle = M_\mu^2\,|\mu\rangle$$

- Since $\mathcal{C}$ is hermitian, the mass square is real:

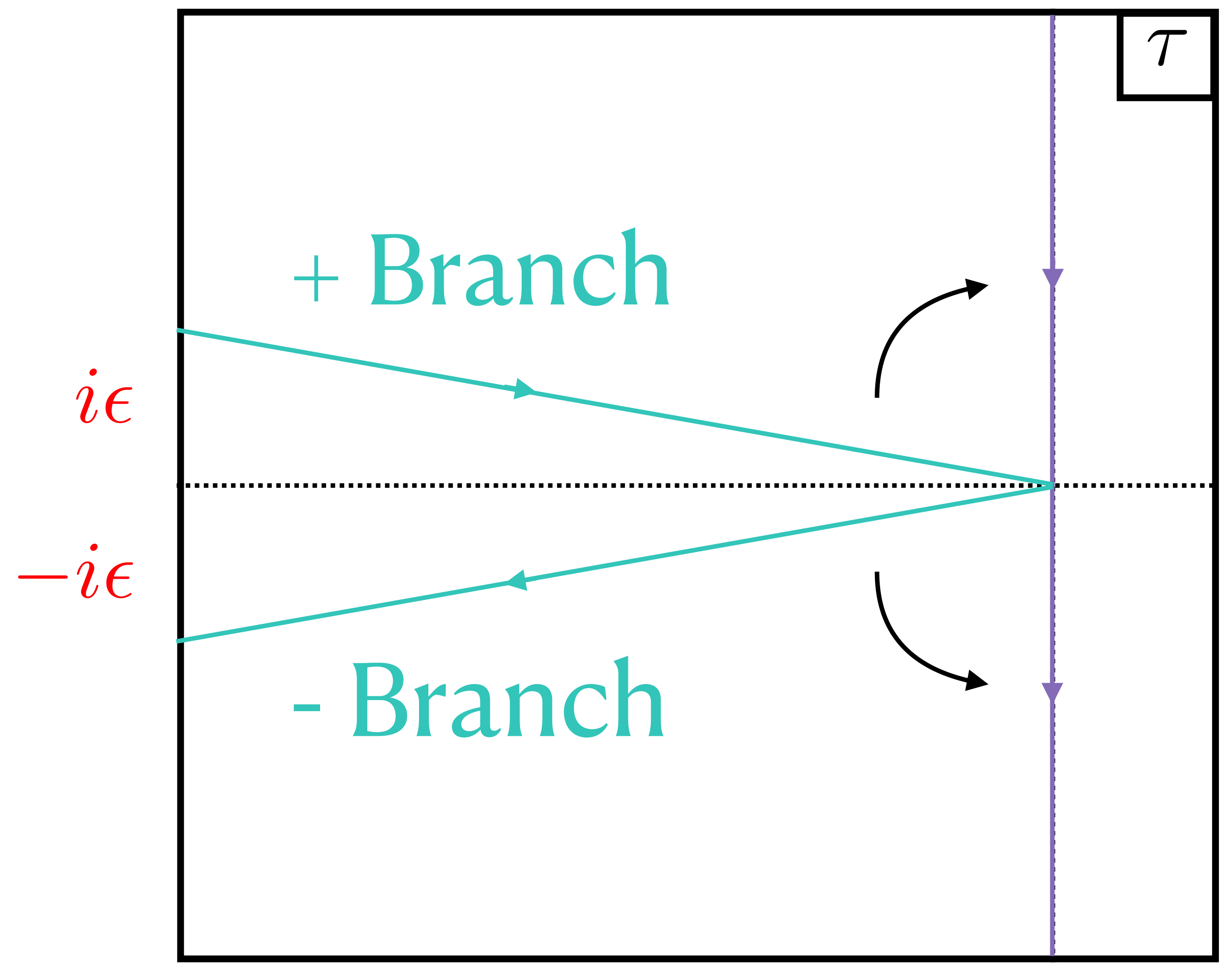
$$M_\mu^2 \in \mathbb{R}$$ Like in Minkowski space!

- Unlike flat space, the relevant parameter is not the mass but the dS frequency μ:

$$\frac{M_\mu^2}{H^2} = \mu^2 + \frac{d^2}{4}$$

- This divides the possible irreps among three series.

Finding Harmonic Functions



Schwinger-Keldysh (or in-in) contour

$\mathcal{C}$ is only self-adjoint on the Wick rotated axis $z = e^{\mp\frac{i\pi}{2}}\tau$

$\mathrm{dS}_{d+1} \to \mathrm{EAdS}_{d+1}$ and

$$\Box_{\mathrm{dS}} \to \Box_{\mathrm{EAdS}} = z^2\partial_z^2 - (d-1)z\partial_z + z^2\partial_i^2$$

We define the harmonic function as:

$$\begin{cases} H^2\Box_{\mathrm{EAdS}}\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{p}) & = & -M_\mu^2\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{p}) \\ \partial_i\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{p}) & = & -ip_i\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{p}) \end{cases} \qquad \frac{M_\mu^2}{H^2} = \mu^2 + \frac{d^2}{4}$$

Solution consistent with Bunch-Davies vacuum boundary conditions:

$$\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{p}) = e^{-i\boldsymbol{x}\cdot\boldsymbol{p}}\frac{H^{\frac{d+1}{2}}}{\sqrt{\pi}}z^{\frac{d}{d}}K_{i\mu}(pz)$$

Modified Bessel function

Kontorovich-Lebedev-Fourier Space

The harmonic function defines the Kontorovich-Lebedev-Fourier (or KLF) integral transform:

$$\boxed{\begin{aligned} f(z,\boldsymbol{x}) &= \int_{\mathrm{KLF}} \Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{x}) f^{(\mu)}_{\boldsymbol{p}} \\ f^{(\mu)}_{\boldsymbol{p}} &= \int_{\mathrm{EAdS}} \left[\Phi^{(\mu)}_{\boldsymbol{p}}(z,\boldsymbol{x})\right]^* f(z,\boldsymbol{x}) \end{aligned}}$$

with

$$\int_{\mathrm{EAdS}} \equiv \int_0^\infty \frac{\mathrm{d}z}{(Hz)^{d+1}} \int_{\mathbb{R}^d} \mathrm{d}^d \mathrm{x}\,.$$

$$\int_{\mathrm{KLF}} \equiv \int_{-\infty}^{+\infty} \mathrm{d}\mu\, \mathcal{N}_\mu \int_{\mathbb{R}^d} \frac{\mathrm{d}^d \boldsymbol{p}}{(2\pi)^d}\,, \quad \mathcal{N}_\mu = \frac{\mu}{\pi}\sinh(\pi\mu)$$

This is the on-shell expansion. Contrary to flat space, there is no equivalent off-shell expansion.

Important: The KLF integration only runs over the principal series!

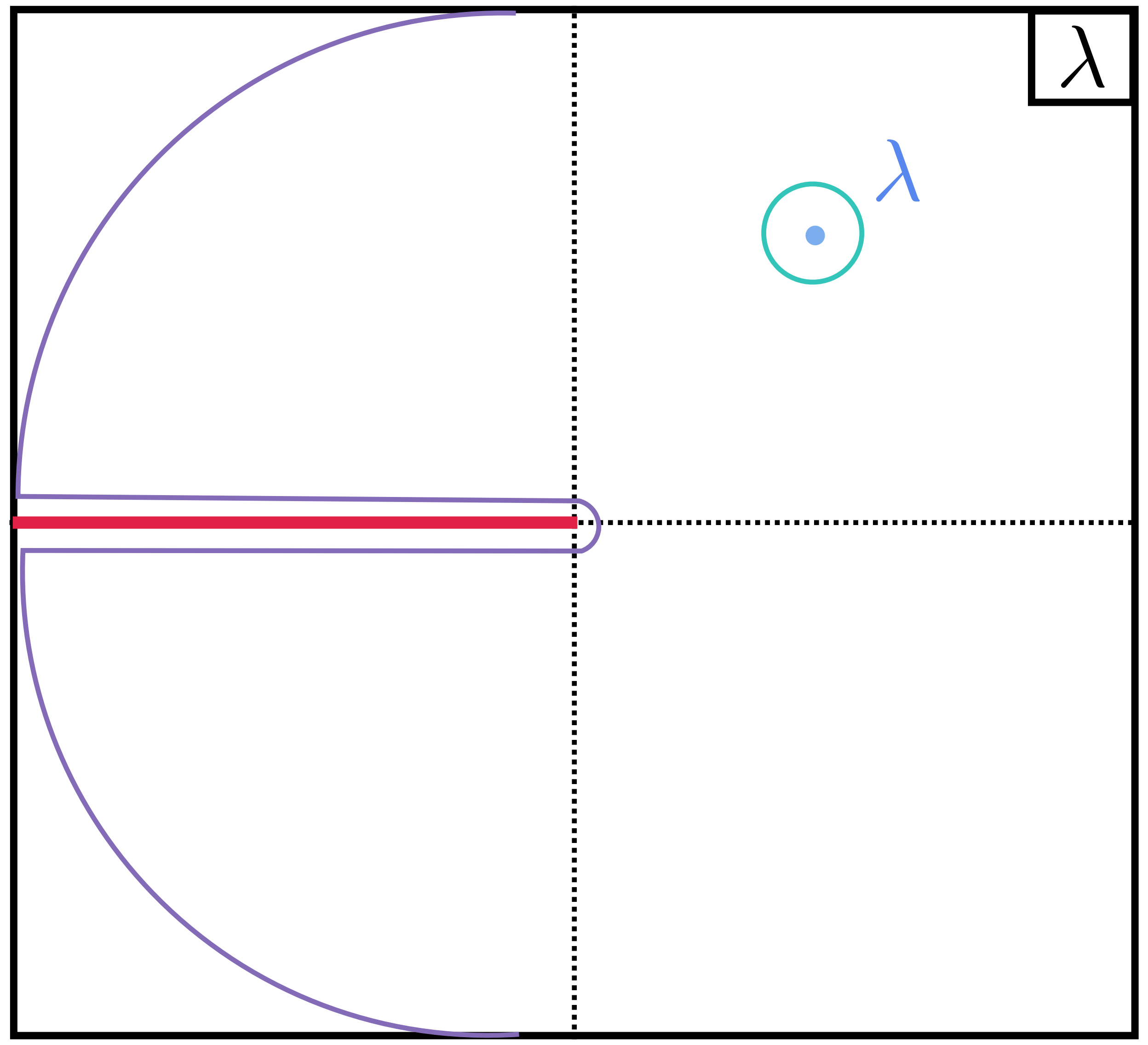
Let us see why!

Completeness of the Principal Series (1/2)

Consider the Green function of the shifted Euclidean Laplacian in spatial Fourier space:

$$\Delta_z = z^{d+1}\frac{\partial}{\partial z}\left(\frac{1}{z^{d-1}}\frac{\partial}{\partial z}\right) - (pz)^2 + \frac{d^2}{4} \qquad \text{and} \qquad (\Delta_z - \lambda)G^\lambda(z,z') = z'^{d+1}\,\delta(z-z')$$

From homogeneous solution and matching conditions, we get



$$G^\lambda(z,z') = -(zz')^{\frac{d}{2}}\begin{cases} I_{\sqrt{\lambda}}(pz)K_{\sqrt{\lambda}}(pz') & \text{for } z < z' \\ I_{\sqrt{\lambda}}(pz')K_{\sqrt{\lambda}}(pz) & \text{for } z' < z \end{cases}$$

Analytic in the cut plane $\lambda \in \mathbb{C} \setminus (-\infty, 0)$

$$G^\lambda(z,z') = \int_{-\infty}^{0} \frac{\mathrm{d}\lambda'}{2\pi i}\,\frac{\mathrm{Disc}_{\lambda'}(G^{\lambda'}(z,z'))}{\lambda' - \lambda}$$

Completeness of the Principal Series (2/2)

Consider the Green function of the shifted Euclidean Laplacian in spatial Fourier space:

$$\Delta_z = z^{d+1}\frac{\partial}{\partial z}\left(\frac{1}{z^{d-1}}\frac{\partial}{\partial z}\right) - (pz)^2 + \frac{d^2}{4} \quad \text{and} \quad (\Delta_z - \lambda)G^\lambda(z,z') = z'^{d+1}\,\delta(z-z')$$

Using Bessel function connection formula and set $\lambda = -\mu^2$, we get

$$G^\lambda(z,z') = -\frac{(zz')^{\frac{d}{2}}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\mu\,\mathcal{N}_\mu \frac{K_{i\mu}(pz)K_{i\mu}(pz')}{\mu^2-\mu_\lambda^2}$$

Acting with $\Delta_z - \lambda$ gives the completeness relation among the function $K_{i\mu}$:

$$\frac{(zz')^{\frac{d}{2}}}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\mu\,\mathcal{N}_\mu\, K_{i\mu}(kz)K_{i\mu}(kz') = z^{d+1}\delta(z-z')$$

Where is the complementary Series?

- We just showed that: $$\boxed{f(z,\boldsymbol{x})} = \int_{\boxed{\mathrm{KLF}}} \Phi_{\boldsymbol{p}}^{(\mu)}(z,\boldsymbol{x}) f_{\boldsymbol{p}}^{(\mu)}$$

$\mathrm{L}^2\,[\mathrm{EAdS}_{d+1}]$

Principal series only

- We restricted to L^2 functions since they are those that appear in the path integral!
- Correlators can be more general functions and generally include all the irreducible representations of $\mathrm{SO}(1,d+1)$.
- For instance, Källén-Lehmann representation of a generic two-point function:

$$G_{\mathcal{O}}(X_1^E, X_2^E) = \int_{\mathcal{P}\oplus\mathcal{C}} \mathrm{d}\mu \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d} \boxed{\rho_{\mathcal{O}}^{\mathcal{P},\mathcal{C}}(\mu)} \Phi_{\boldsymbol{p}}^{(\mu)}(z_1,\boldsymbol{x}_1)\Phi_{-\boldsymbol{p}}^{(\mu)}(z_2,\boldsymbol{x}_2)$$

with

$$X^E = (z,\boldsymbol{x})$$

Positive spectral densities

- The principal series is enough to determine the other contributions.

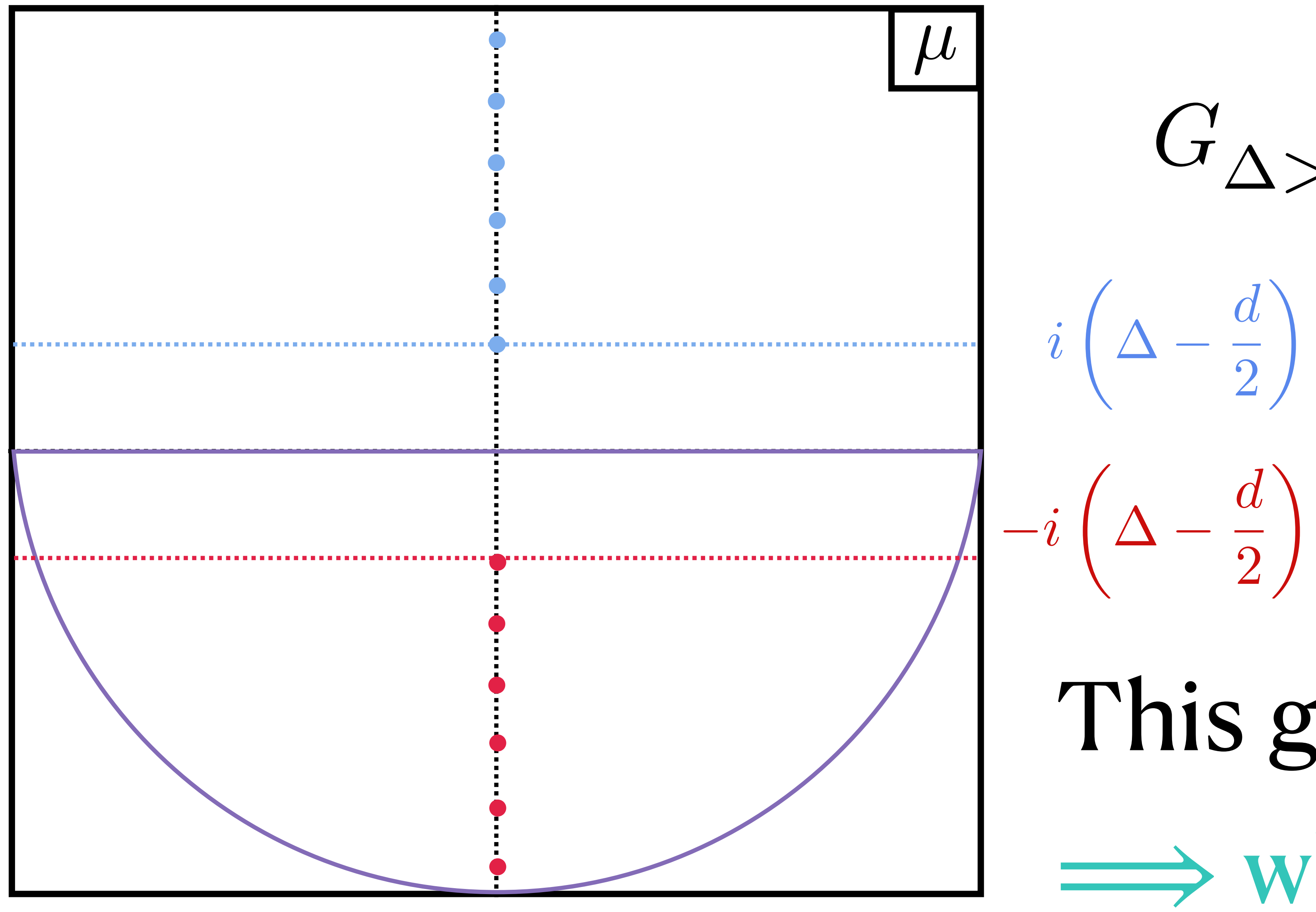
Example: CFT 2-point function (1/3)

- For a bulk scalar CFT, the non-perturbative 2-point function is:

$$G_\Delta(X_1^E, X_2^E) = \frac{1}{2^\Delta(1 - \underbrace{\boxed{X_1^E \cdot X_2^E}}_{\in(-\infty,-1)})^\Delta} \quad \text{with} \quad \Delta > \frac{d-1}{2}$$

being the operator scaling dimension

- The KLF transform converges for $\Delta > \dfrac{d}{2}$ (= regime of square integrability!)



$$G_{\Delta > \frac{d}{2}}(X_1^E, X_2^E) = c_\Delta \int_{\mathrm{KLF}} \Phi_{\boldsymbol{p}}^{(\mu)}(X_1^E)\Phi_{-\boldsymbol{p}}^{(\mu)}(X_2^E)\Gamma\left(\Delta - \frac{d}{2} \pm i\mu\right)$$

$$= 2\pi i \sum_{n=0}^{\infty} \mathrm{Res}\,(\bullet)$$

This gives a series expansion also valid for $\Delta < \dfrac{d}{2}$

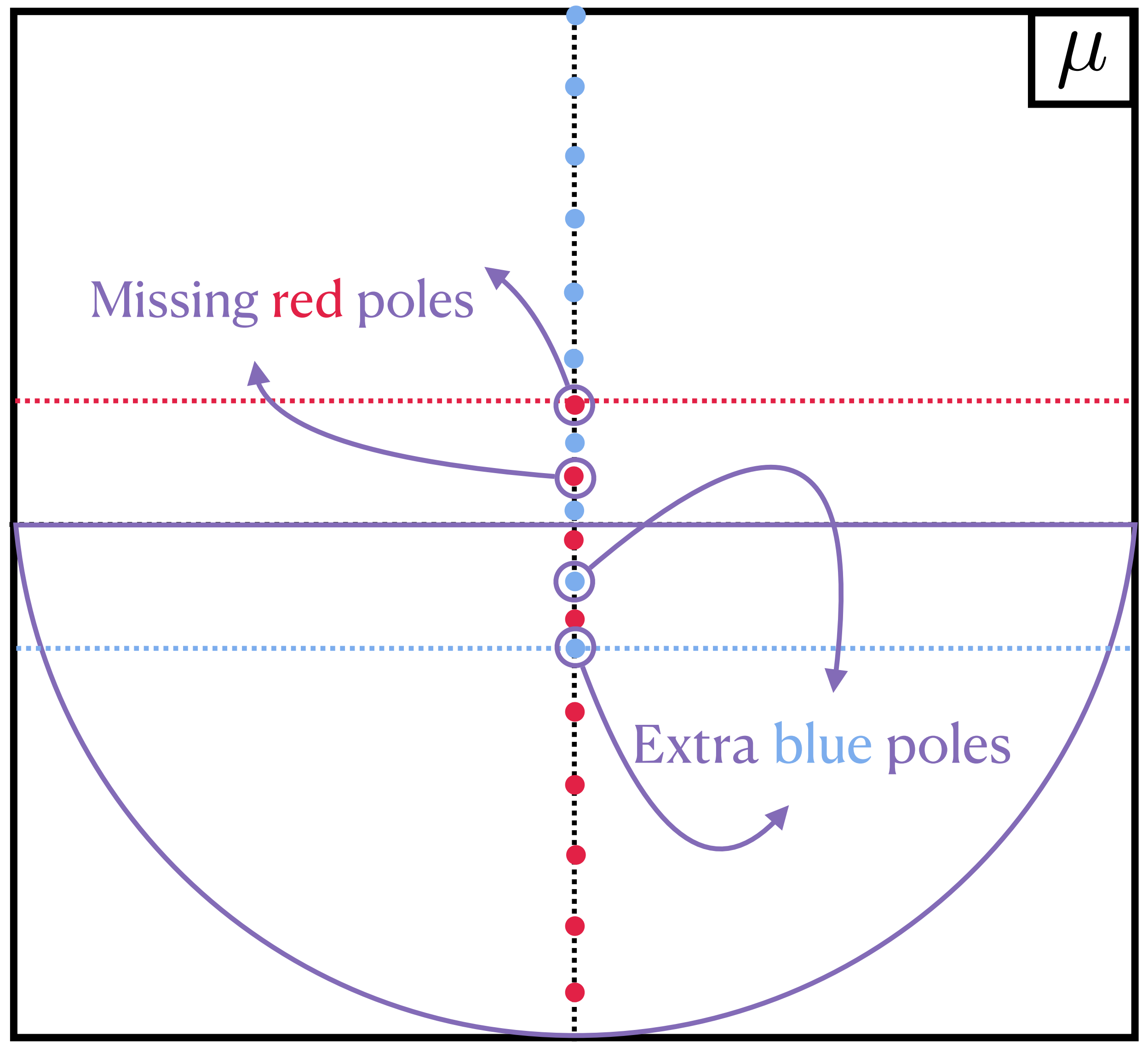
$\Longrightarrow$ what happens in this regime?

Example: CFT 2-point function (2/3)

- For a bulk scalar CFT, the non-perturbative 2-point function is:

$$G_\Delta(X_1^E, X_2^E) = \frac{1}{2^\Delta(1-\underbrace{\boxed{X_1^E\cdot X_2^E}}_{\in(-\infty,-1)})^\Delta} = 2\pi i \sum_{n=0}^{\infty} \mathrm{Res}(\color{red}{\bullet})$$ (This is valid for all Δ)

- For $\Delta < \frac{d}{2}$ this is not given by the principal integral anymore!



$$c_\Delta \int_{\mathrm{KLF}} \Phi_{\boldsymbol{p}}^{(\mu)}(X_1^E)\Phi_{-\boldsymbol{p}}^{(\mu)}(X_2^E)\Gamma\left(\Delta - \frac{d}{2} \pm i\mu\right)$$

$$= 2\pi i \sum_{n=N_*}^{\infty} \mathrm{Res}(\color{red}{\bullet}) + 2\pi i \sum_{n=0}^{N_*-1} \mathrm{Res}(\color{blue}{\bullet})$$

$$= G_{\Delta<\frac{d}{2}}(X_1^E, X_2^E) - \boxed{2\pi i \sum_{n=0}^{N_*-1}\left[\mathrm{Res}(\color{red}{\bullet}) - \mathrm{Res}(\color{blue}{\bullet})\right]}$$

Complementary series!

Example: CFT 2-point function (3/3)

- For unitary CFT, $\Delta > \dfrac{d-1}{2}$ there can only be a single contribution.
- We recover the result from arXiv:2306.00090 from a KLF point of view:

$$\boxed{\begin{aligned} G_\Delta(X_1^E, X_2^E) = &\int_{\mathrm{KLF}} \Phi_{\boldsymbol{p}}^{(\mu)}(X_1^E)\Phi_{-\boldsymbol{p}}^{(\mu)}(X_2^E)\rho_{\mathcal{O}}^{\mathcal{P}}(\mu) \\ &+\Theta\left(\frac{d}{2}-\Delta\right) 4\pi i \underset{\mu=i\left(\Delta-\frac{d}{2}\right)}{\mathrm{Res}} \left(\rho_{\mathcal{O}}^{\mathcal{P}}(\mu)\right) \int \frac{\mathrm{d}^d\boldsymbol{p}}{(2\pi)^d} \Phi_{\boldsymbol{p}}^{\left(\Delta-\frac{d}{2}\right)}(X_1^E)\Phi_{-\boldsymbol{p}}^{\left(\Delta-\frac{d}{2}\right)}(X_2^E) \end{aligned}}$$

- The complementary series contributions in the spectral decompositions can be read from the non-analyticities of $\rho_{\mathcal{O}}^{\mathcal{P}}(\mu)$ along the imaginary axis.

No known examples!!

Branch cuts $\rightarrow$ continuous contributions

Poles $\rightarrow$ isolated contributions

Extending KLF

- Isolated complementary contributions can be accounted for a generalised δ function:

$$\int_{\mathrm{EAdS}} \left[\Phi_{\boldsymbol{p}}^{(\mu)}(z,\boldsymbol{x})\right]^* \Phi_{\boldsymbol{p}'}^{(\alpha)}(z,\boldsymbol{x}) = \frac{(2\pi)^d}{\mathcal{N}_\mu}\delta^{(d)}(\boldsymbol{p}-\boldsymbol{p}')\hat{\delta}_\alpha(\mu)$$

For any $\alpha \in \mathbb{C}$

It acts on square-integrable functions as a shadow symmetric analytical continuation:

$$\int_{-\infty}^{\infty} \mathrm{d}\mu f_{\boldsymbol{p}}^{(\mu)}\hat{\delta}_\alpha(\mu) = \frac{1}{2}\left(f_{\boldsymbol{p}}^{(\alpha)} + f_{\boldsymbol{p}}^{(-\alpha)}\right)$$

- This allows to expand more general functions in KLF:

$$G_\Delta(X_1^E, X_2^E) = \int_{\mathrm{KLF}} \frac{\boxed{\rho_{\mathcal{O}}(\mu)}}{\mathcal{N}_\mu}\Phi_{\boldsymbol{p}}^{(\mu)}(X_1^E)\Phi_{-\boldsymbol{p}}^{(\mu)}(X_2^E)$$

$$\boxed{\rho_{\mathcal{O}}(\mu) = \rho_{\mathcal{O}}^{\mathcal{P}}(\mu) + 4\pi i\Theta\left(\frac{d}{2}-\Delta\right)\hat{\delta}_{i\left(\Delta-\frac{d}{2}\right)}(\mu) \underset{\mu=i\left(\Delta-\frac{d}{2}\right)}{\mathrm{Res}}\left(\rho_{\mathcal{O}}^{\mathcal{P}}(\mu)\right)}$$

The KLF coefficient is now a distribution!

Outline

- Warm up: Minkowski Momentum Space
  - Review of the construction in Minkowski space is a dS-like way
- de Sitter Momentum Space
  - Construction of the Kontorovich-Lebedev-Fourier space (KLF)
  - Example of the conformal two point function
- QFT in momentum space: Perturbation Theory
  - Path integral in KLF space
  - Free theory and KLF propagators
  - Interaction vertices and examples: single exchange & loop integral

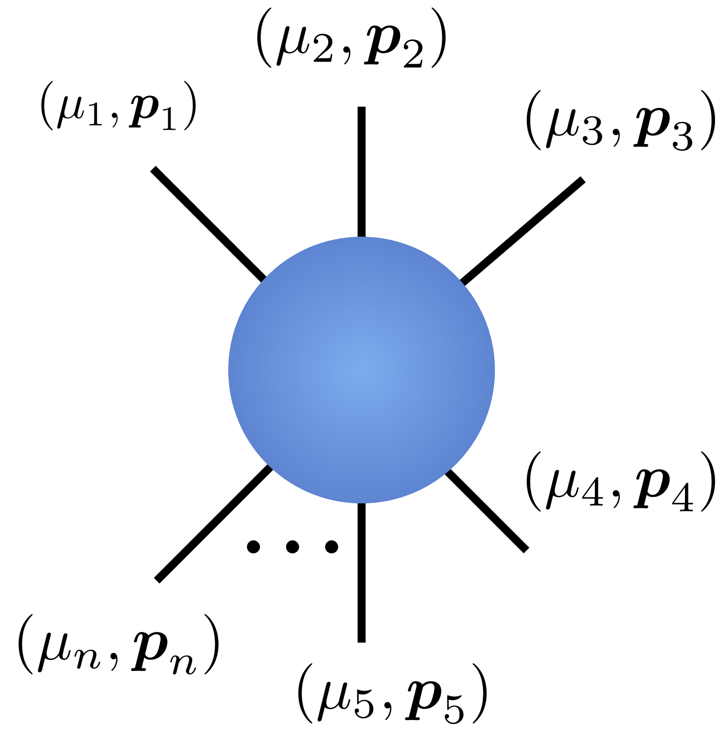
KLF Correlators

- The path integral is unaffected by the functional basis we use→ we write it using KLF:

$$Z[J_+, J_-] = \int \mathcal{D}\varphi_\pm e^{iS_+[\varphi_+]-iS_-[\varphi_-]+\int_{\mathrm{KLF}}(\varphi_+ J_+ + \varphi_- J_-)}$$

- This motivates the definition of KLF correlators:

$$\mathcal{G}_{\mathrm{a}_n\ldots\mathrm{a}_n}\begin{pmatrix}\mu_1\ldots\mu_n\\ \boldsymbol{p}_1\ldots\boldsymbol{p}_n\end{pmatrix} = \prod_{i=1}^{n}\frac{(2\pi)^d}{\mathcal{N}_{\mu_i}}\frac{\delta Z[J_+, J_-]}{\delta J^{\mu_i}_{\mathrm{a}_i,\boldsymbol{p}_i}}\bigg|_{J_\pm=0} =$$



- Related to real space correlators by a KLF inverse transform:

$$G_{\mathrm{a}_1\ldots\mathrm{a}_n}(X^E_1\ldots X^E_n) = \int_{\mathrm{KLF}^n}\prod_{j=1}^{n}\Phi^{(\mu_j)}_{\boldsymbol{p}_j}(X^E_j)\mathcal{G}_{\mathrm{a}_n\ldots\mathrm{a}_n}\begin{pmatrix}\mu_1\ldots\mu_n\\ \boldsymbol{p}_1\ldots\boldsymbol{p}_n\end{pmatrix}$$

- As in the CFT example, $\mathcal{G}_{\mathrm{a}_n\ldots\mathrm{a}_n}\begin{pmatrix}\mu_1\ldots\mu_n\\ \boldsymbol{p}_1\ldots\boldsymbol{p}_n\end{pmatrix}$ can be a distribution.

e.g.

$$\mathcal{G}_{\mathrm{a}_n\ldots\mathrm{a}_n}\begin{pmatrix}\mu_1\ldots\mu_n\\ \boldsymbol{p}_1\ldots\boldsymbol{p}_n\end{pmatrix} \sim \hat{\delta}_\alpha(\mu_\ell)\times\#$$

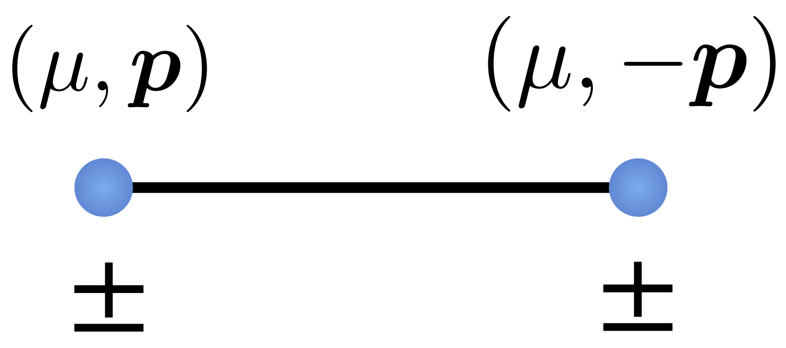
Free Theory

- A free field in dS has the following action:

$$\pm i S_{\pm}^{\text{free}}[\varphi] = \pm\frac{i}{2}\int_{-\infty(1\mp i\epsilon)}^{0} \frac{\mathrm{d}\tau\,\mathrm{d}^d\boldsymbol{x}}{(-H\tau)^{d-1}}\left[(\partial_\tau\varphi)^2-(\partial_i\varphi)^2-\frac{m_\varphi^2}{H^2\tau^2}\varphi^2\right] = \frac{H^2 e^{\pm\frac{i(d-1)\pi}{2}}}{2}\int_{\text{KLF}} \varphi_{\boldsymbol{p}}^{(\mu)}(\mu^2-\mu_\varphi^2)\varphi_{-\boldsymbol{p}}^{(\mu)}$$

- One can perform the Gaussian path integral exactly (Weinberg 2005 in real space).
- This gives the following KLF-space Schwinger-Keldysh propagators:

$$\boldsymbol{\mathcal{G}}_{\mu_\varphi}\begin{pmatrix}\mu\ \mu' \\ \boldsymbol{p},\boldsymbol{p}'\end{pmatrix} = \frac{(2\pi)^d}{\mathcal{N}_\mu}\hat{\delta}_{\mu'}(\mu)\delta^d(\boldsymbol{p}+\boldsymbol{p}') \begin{array}{c} \begin{array}{cc} \quad + & \qquad\qquad - \end{array} \\ \begin{pmatrix} \frac{e^{-\frac{i\pi(d-1)}{2}}}{(\mu^2-\mu_\varphi^2)_{i\epsilon}} & \frac{\hat{\delta}_{\mu_\varphi}(\mu)}{\mathcal{N}_\mu} \\ \frac{\hat{\delta}_{\mu_\varphi}(\mu)}{\mathcal{N}_\mu} & \frac{e^{+\frac{i\pi(d-1)}{2}}}{(\mu^2-\mu_\varphi^2)_{-i\epsilon}} \end{pmatrix} \end{array} \begin{array}{c} + \\ \\ - \end{array} =$$



where the dS $i\epsilon$ prescription is:

$$\frac{1}{(\mu^2-\mu_\varphi^2)_{i\epsilon}} \equiv \frac{1}{2\sinh(\pi\mu_\varphi)}\left[\frac{e^{+\pi\mu_\varphi}}{\mu^2-\mu_\varphi^2+i\epsilon}-\frac{e^{-\pi\mu_\varphi}}{\mu^2-\mu_\varphi^2-i\epsilon}\right]$$

see

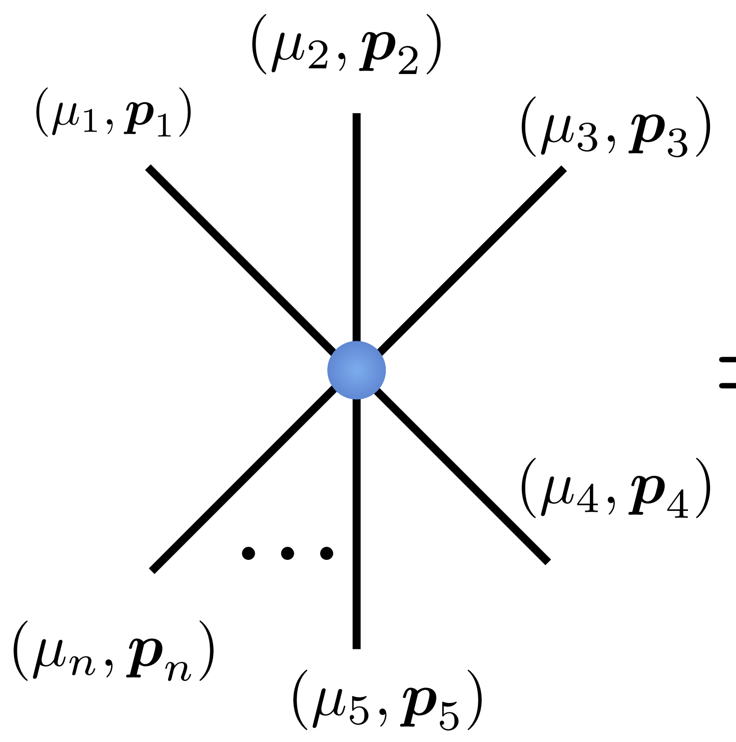
Pimentel,Melville arXiv: 2404.05712

Werth arXiv:2409.02072

Interaction Vertices

- Interactions can be treated perturbatively as in standard QFT.

- For example, polynomial self-interaction $\mathcal{L}_I = -\lambda\varphi^n/n!$:



$$= -i\mathrm{a}\lambda(2\pi)^d\delta^d\left(\sum_{j=1}^{n}\boldsymbol{p}_j\right)\frac{H^{\frac{(n-2)(d+1)}{2}}e^{\mathrm{a}\frac{i\pi d}{2}}}{\pi^{\frac{n}{2}}}\mathcal{I}^{\mu_1\dots\mu_n}_{p_1\dots p_n}$$

where the vertex function $\mathcal{I}^{\mu_1\dots\mu_n}_{p_1\dots p_n}$ is given by the following integral:

$$\mathcal{I}^{\mu_1\dots\mu_n}_{p_1\dots p_n} = \int_0^\infty \mathrm{d}z\; z^{\frac{d(n-2)}{2}-1}\prod_{j=1}^{n}K_{i\mu_j}(p_j z)$$

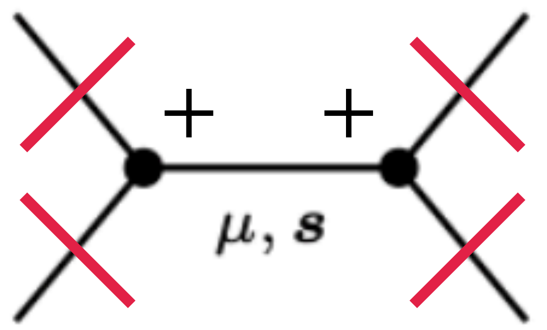
- There is no frequency conservation at the vertices!

This is because of a lack of off-shell expansion

Application: Single exchange Diagram

- Exchange processes naturally appear in a spectral representation:

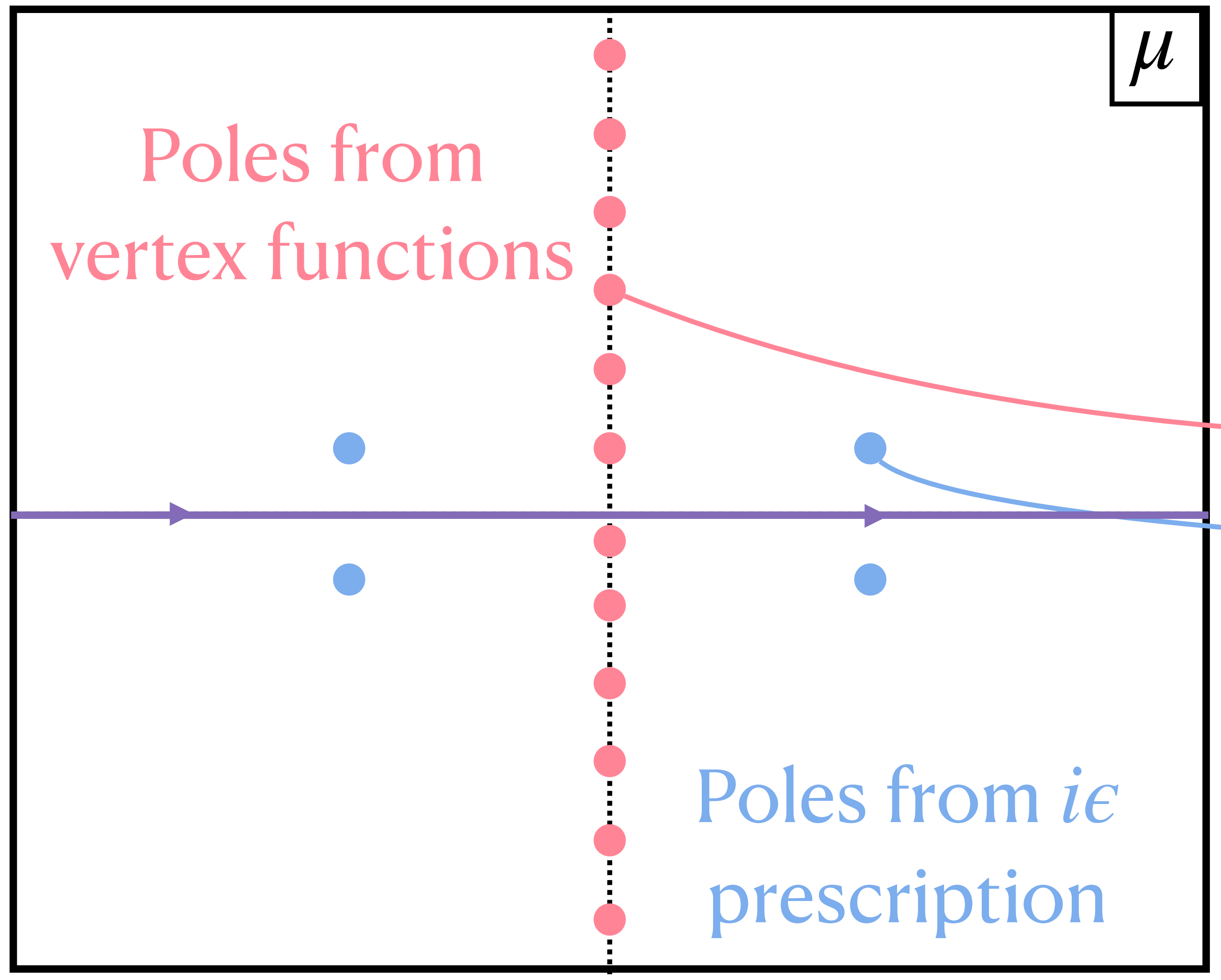
$$\boxed{\tilde{\mathcal{G}}_{++}}\left(\begin{smallmatrix}\mu_1\mu_2\mu_3\mu_4\\ \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4\end{smallmatrix}\right) \equiv$$



$$= (-i\lambda)^2 \frac{H^{d+1}e^{-\frac{i\pi d}{2}}}{\pi^3} \int_{-\infty}^{+\infty} \mathrm{d}\mu \mathcal{N}_\mu \frac{\mathcal{I}^{\mu_1\mu_2\mu}_{k_1k_2s}\,\mathcal{I}^{\mu\mu_3\mu_4}_{sk_3k_4}}{(\mu^2-\mu_\chi^2)_{i\epsilon}}$$

Amputated diagram:
no external legs

- The remaining spectral integral can be performed by closing the contour:



$$\tilde{\mathcal{G}}_{++}\left(\begin{smallmatrix}\mu_1\mu_2\mu_3\mu_4\\ \boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4\end{smallmatrix}\right) \propto u\left(\frac{u}{2}\right)^{d-3}\left[\left(u^{-2}-1\right)\left(v^{-2}-1\right)\right]^{\frac{d-3}{4}}$$

$$\times \sum_{n=0}^{+\infty} \frac{(-1)^n}{\left(n+\frac{d-2}{2}\right)^2+\mu_\chi^2}\left(\frac{u}{v}\right)^n \frac{\Gamma(n+d-2)}{\Gamma(n+1)}\, {}_2F_1\left[\begin{matrix}\frac{d-2+n}{2}, \frac{d-1+n}{2}\\ \frac{d}{2}+n\end{matrix}; u^2\right] {}_2F_1\left[\begin{matrix}\frac{1-n}{2}, -\frac{n}{2}\\ \frac{4-d}{2}-n\end{matrix}; v^2\right]$$

$$-\frac{1}{2}\left|\Gamma\left(\frac{d-2}{2}+i\mu_\chi\right)\right|^4 P^{-\frac{d-3}{2}}_{i\mu_\chi-1/2}\left(e^{+i\pi}u^{-1}\right) P^{-\frac{d-3}{2}}_{i\mu_\chi-1/2}\left(v^{-1}\right), \qquad u, v \equiv s/k_{12,34}$$

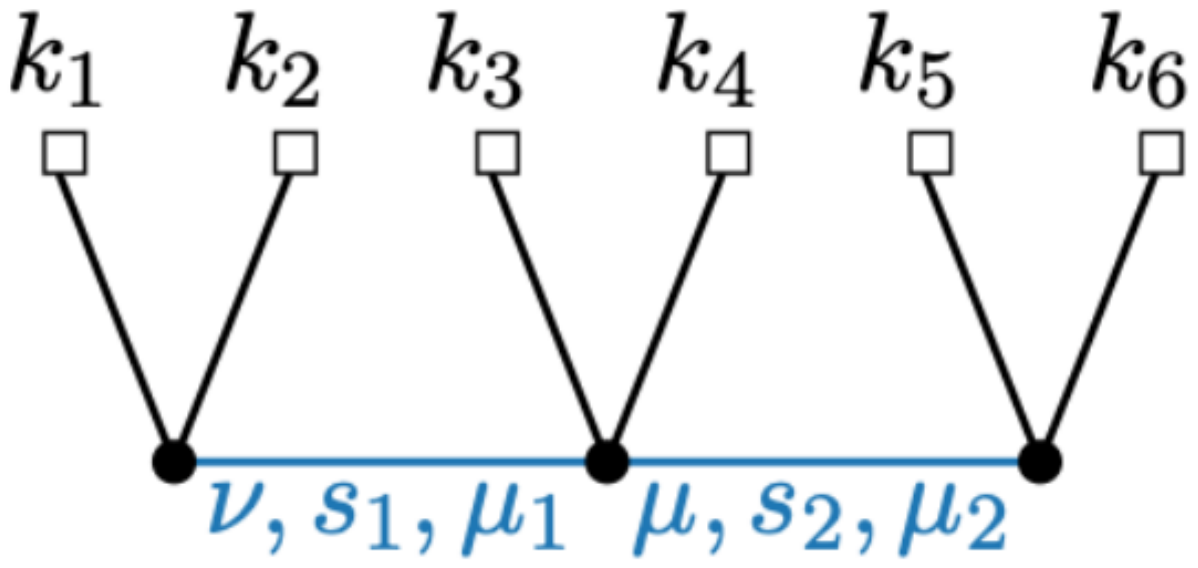
Computation by Nathan Belrhali in general d, working on double exchange.

Generalises arXiv:2409.02072

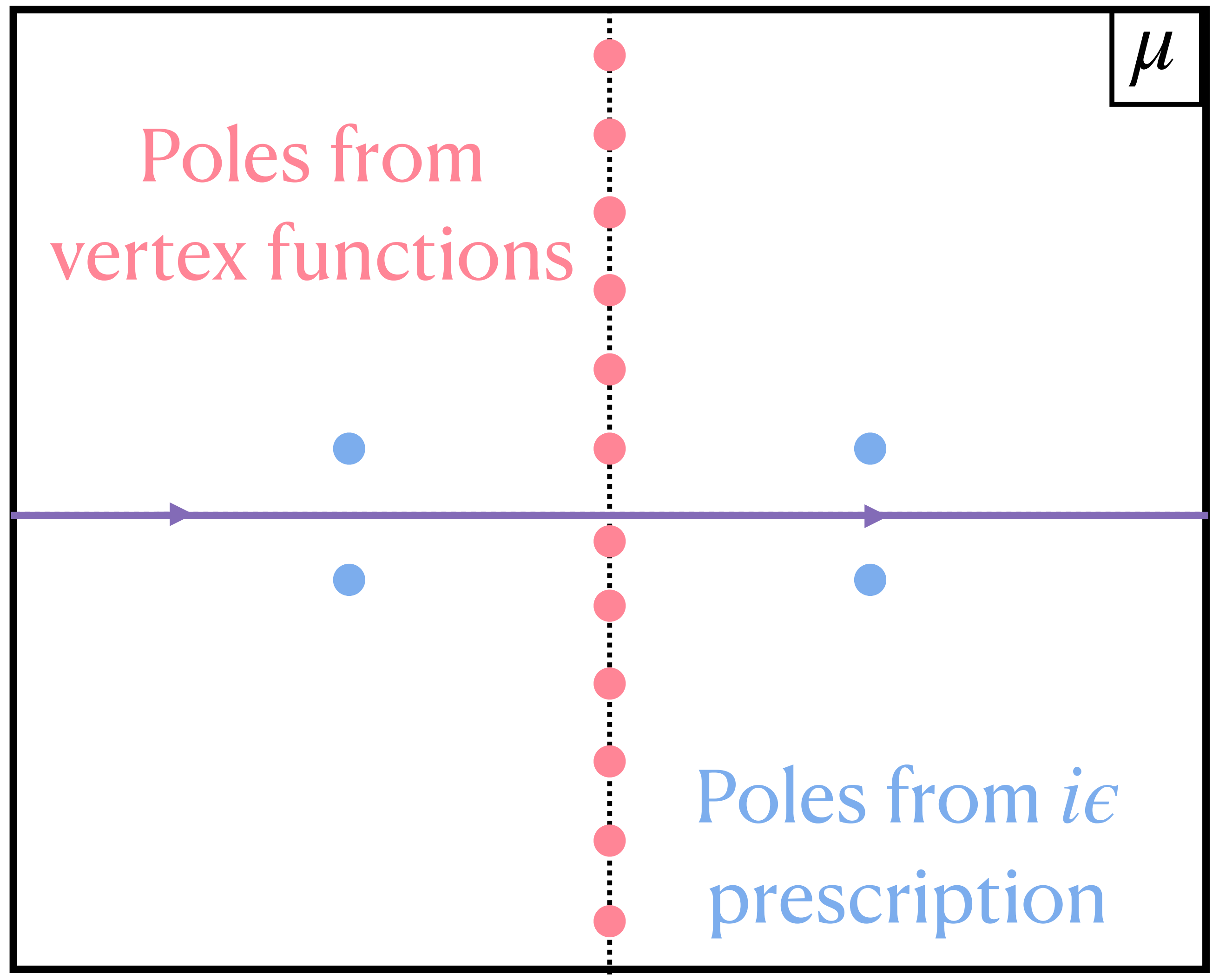
Application: Double exchange Diagram

- Exchange processes naturally appear in a spectral representation:

$$G_{+++} \equiv$$



$$= \int_{-\infty}^{+\infty} \mathrm{d}\nu\, \mathcal{N}(\nu) \int_{-\infty}^{+\infty} \mathrm{d}\mu\, \mathcal{N}(\mu)\, \frac{\mathcal{I}^{\left(\frac{i}{2},\frac{i}{2},\nu\right)}_{k_1,k_2,s_1}\, \mathcal{I}^{\left(\frac{i}{2},\frac{i}{2},\nu,\mu\right)}_{k_3,k_4,s_1,s_2}\, \mathcal{I}^{\left(\frac{i}{2},\frac{i}{2},\mu\right)}_{k_5,k_6,s_2}}{\left(\nu^2-\mu_1^2\right)_{i\epsilon}\left(\mu^2-\mu_2^2\right)_{i\epsilon}}$$



- Two layers of integration but it works the same way!
- Each layer comes with with two contributions from the two types of poles.

$$G_{+++} = \boxed{G^{\mathcal{I},\mathcal{I}}_{+++}} + G^{\mathcal{I},P_\epsilon}_{+++} + G^{P_\epsilon,\mathcal{I}}_{+++} + G^{P_\epsilon,P_\epsilon}_{+++}$$

The most difficult one!

Application: Double exchange Diagram

- First original KLF computation by Nathan Belrhali:

- Produces the background signal.

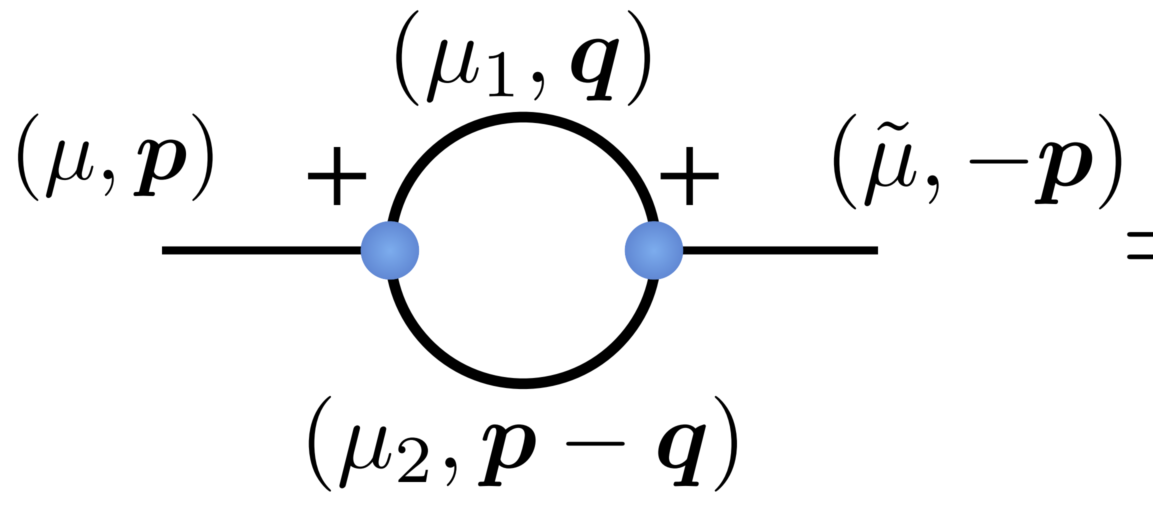
- The three other contributions are simpler.

$$
\begin{aligned}
G^{\mathcal{I},\mathcal{I}}_{+++} &= \frac{\pi^5(-1)^d 2^{d-3/2}}{\sqrt{k_1k_2k_3k_4 s_1^{\frac{d-2}{2}} s_2^{d-1}}} \left(\frac{s_2}{s_1}\right)^{\frac{3d-4}{2}} \left[\left(\frac{k_{12}}{s_1}\right)^2 - 1\right]^{\frac{d-3}{4}} \left[\left(\frac{k_{56}}{s_2}\right)^2 - 1\right]^{\frac{d-3}{4}} \\
&\times \sum_{n=0}^{+\infty}\sum_{p=0}^{+\infty} \frac{(-1)^p}{n!p!} \left(\frac{2k_{34}}{s_1}\right)^n \left(\frac{s_2}{s_1}\right)^p \left(\frac{s_2}{k_{56}}\right)^p \frac{p+n+\frac{3d-4}{2}}{\left(p+n+\frac{3d-4}{2}\right)^2+\mu_1^2} \frac{p+\frac{d-2}{2}}{\left(p+\frac{d-2}{2}\right)^2+\mu_2^2} \\
&\times Q^{\frac{d-3}{2}}_{p+n+\frac{3d-5}{2}}\left(\frac{k_{12}}{s_1}\right) \mathcal{F}_4\left(\begin{matrix} -\frac{n}{2} & , & \frac{1-n}{2} \\ 3-p-n-\frac{3d}{2} & , & p+\frac{d}{2} \end{matrix} ; \left(\frac{s_1}{k_{34}}\right)^2, \left(\frac{s_2}{k_{34}}\right)^2\right) \\
&\times {}_2\mathcal{F}_1\left(\begin{matrix} -\frac{p}{2}, \frac{1-p}{2} \\ 2-\frac{d}{2}-p \end{matrix} ; \left(\frac{s_2}{k_{56}}\right)^2\right) \\
&+ \frac{\pi^{7/2}2^{d-3/2}e^{i\pi\frac{d}{2}}}{\sqrt{k_1k_2k_3k_4 s_1^{\frac{d-2}{2}} s_2^{d-1}}} \left[\left(\frac{k_{12}}{s_1}\right)^2 - 1\right]^{\frac{d-3}{4}} \sum_{p=0}^{+\infty}\sum_{n=0}^{2p+1} \frac{(-1)^{p+1}}{n!} \left(\frac{2k_{34}}{s_1}\right)^n \left(\frac{s_2}{s_1}\right)^{p+1} \\
&\times \frac{(p+1)}{(p+1)^2+\mu_1^2} \frac{n-p+d-2}{(n-p+d-2)^2+\mu_2^2} Q^{\frac{d-3}{2}}_{p+\frac{1}{2}}\left(\frac{k_{12}}{s_1}\right) G_n\left(i(p+1)\right) Q^{\frac{d-3}{2}}_{n-p+d-\frac{5}{2}}\left(\frac{k_{56}}{s_2}\right)
\end{aligned}
\tag{2.5}
$$

where ${}_2\mathcal{F}_1$ and $\mathcal{F}_4$ are respectively regularized Gauss hypergeometric and regularized Appell functions:

Another example: Loop Diagrams

- We write the KLF space loop correction to the propagator:



$$= \frac{\lambda^2 H^{d-3}}{\pi^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu_1 \mathcal{N}_{\mu_1} \mathrm{d}\mu_2 \mathcal{N}_{\mu_2}}{(\mu_1^2 - \mu_\varphi^2)_{i\epsilon} (\mu_2^2 - \mu_\varphi^2)_{i\epsilon}} \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \mathcal{I}^{\mu\mu_1\mu_2}_{pq|\boldsymbol{p}-\boldsymbol{q}|} \mathcal{I}^{\tilde{\mu}\mu_1\mu_2}_{pq|\boldsymbol{p}-\boldsymbol{q}|}$$

- Cubic vertex function is related to the dS Wigner 3μ-symbols:

$$\mathcal{I}^{\mu_1\mu_2\mu}_{p_1p_2p} \sim \boxed{\sqrt{\rho_{\mu_1\mu_2}(\mu)}} \boxed{\begin{pmatrix} \mu_1 & \mu_2 & \mu \\ p_1 & p_2 & p \end{pmatrix}^*}$$

$$(\langle \mu_1, \boldsymbol{p}_1 | \otimes \langle \mu_2, \boldsymbol{p}_2 |) \, | \mu, \boldsymbol{p} \rangle$$

Decomposition of the tensor products

$$\langle \varphi_1\varphi_2(X_1^E) \varphi_1\varphi_2(X_2^E) \rangle = \int_{\mathrm{KLF}} \frac{\rho_{\mu_1\mu_2}(\mu)}{\mathcal{N}_\mu} \Phi^{(\mu)}_{\boldsymbol{k}}(X_1^E) \Phi^{(\mu)}_{-\boldsymbol{k}}(X_2^E)$$

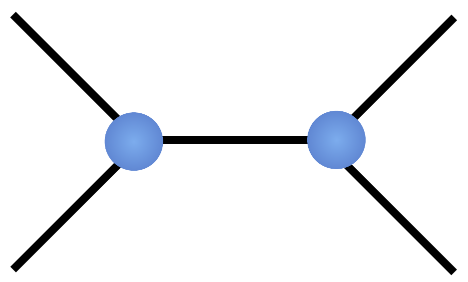
Källén-Lehmann spectral density of the field product

- The momentum integral is fully fixed by kinematic:

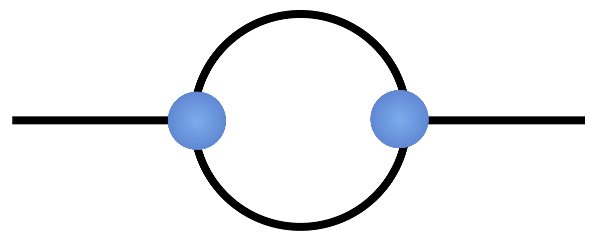
$$\frac{H^{d-3}}{\pi^3} \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \mathcal{I}^{\mu\mu_1\mu_2}_{pq|\boldsymbol{p}-q|} \mathcal{I}^{\tilde{\mu}\mu_1\mu_2}_{pq|\boldsymbol{p}-q|} = \frac{\rho_{\mu_1\mu_2}(\mu)}{H^4 \mathcal{N}_\mu^2} \hat{\delta}_{\tilde{\mu}}(\mu)$$

Conclusion and prospects

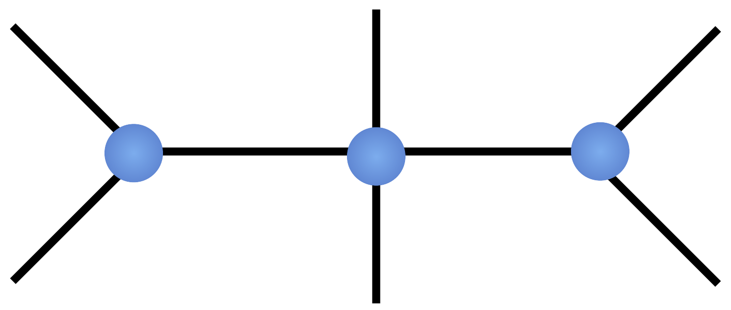
- We found the momentum space adapted to dS isometries: KLF space.
- Perturbative computations become transparent in this picture:



Single exchange:
done



Loop corrections:
almost done



Double exchange:
work in progress!

Future Directions

- General statement about analytic structure in KLF → Causality?
- Proper language for effective field theory?
- Effect of unitarity → EFT positivity bound?
- Towards renormalisation in dS - KLF Polchinksi equation?